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Scasimir operator, Scentre and Representations of $\mathcal{U}_q(osp(1|2))$

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Abstract

A bosonic operator of $\mathcal{U}_q(osp(1|2))$ that anticommutes with the fermionic generators appears to be useful to describe the relations in the centre of $\mathcal{U}_q(osp(1|2))$ for q a root of unity (in the unrestricted specialisation). As in the classical case, it also simplifies the classification of finite dimensional irreducible representations.

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1 Motivations

The centre of the universal enveloping algebra of a semi simple Lie algebra is a free polynomial algebra that can be described quite explicitly with the Harish-Chandra homomorphism. This construction can be extended to the quantum deformation of a semi simple Lie algebra when the deformation parameter q is not a root of unity [1], leading to deformed Casimir operators.

But when q is a root of unity¹, say $q^l = 1$, the Casimir operators do not exhaust the centre anymore. For example if l is odd, the l^{th} powers of the generators are central. Together with the Casimir operators, they satisfy polynomial relations [2, 3, 4, 5, 6]. Hence from the point of view of algebraic geometry, the centre is a non trivial affine variety. From a physical point of view, it is desirable to know the equations of this variety. This is because to exploit efficiently the symmetries of a model one needs a complete set of commuting observables to label the states. If the symmetry of the model is a quantum group associated to a semi simple Lie algebra, the centre together with a Cartan subalgebra gives such a complete set.

For semi simple superalgebras, there is no strict analogue of the Harish-Chandra construction in general [7]. The centre of the universal enveloping algebra does not have to be a free polynomial algebra. But the Casimir operators can still be extended to the quantum deformation and exhaust the centre when q is not a root of unity.

When q is a root of unity, the discussion of the Lie algebra case applies.

The relations in the centre of a quantum deformation of a semi simple (Lie or super) algebra have been computed only for particular examples [4, 5, 6, 8]. All the known cases seem to share some nice features that we briefly recall in section 7.

The study of $\mathcal{U}_q(osp(1|2))$ that we present below has led us to reconsider those general features. In fact, trying to conciliate them with the case at hand, we were led to use the following simple structure: the Casimir operator of $\mathcal{U}_q(osp(1|2))$ [9] is a perfect square, and the square root, despite its bosonic character, anti-commutes with fermions and commutes with bosons, so we decided to call it a Scasimir operator. This operator, first written in [10] is the q -deformation of a classical operator introduced in [11, 12]. The previously observed general features of the relations in the centre satisfied by the Casimir operators now involve the Scasimir operator. The possible generalisation to $\mathcal{U}_q(osp(1|2n))$ involves a detailed study of the Scasimir operator in the non deformed case. This study is presented elsewhere [13]. The importance of the Scasimir operator to classify irreducible finite dimensional irreducible representations of $\mathcal{U}_q(osp(1|2))$ is also emphasised.

We note that the existence of periodic irreducible representations (for q a root of unity, in the unrestricted case) imply the existence of primitive ideals that are not annihilators of irreducible quotients of Verma modules (all of them being annihilated by a common finite power of all the raising generators). This differs from the case of classical (non quantum) (super)algebras [14, 15].

¹We work in the unrestricted specialisation, so we do not introduce divided powers of the generators.

2 Notations

In this paper, q is a complex number such that $q^2 \neq 0, 1$ and l is an integer larger than 2. We also need an indeterminate t to build generating functions.

We denote by $q^{1/2}$ a fixed square root of q and by q' the opposite of q . We set $\eta = (q^{1/2} + q^{-1/2})(q - q^{-1})$.

In the following, we shall often be interested in the case when q is a primitive l^{th} -root of unity. We use l' to denote the order of q' and L to denote the smallest even multiple of l (that is, l if l is even and $2l$ if l is odd). The integer l' is $l/2$ if l is twice an odd integer and L otherwise. The map $l \rightarrow l'$ is one to one.

3 Definitions

The algebra $\mathcal{U}_q(\text{osp}(1|2))$ is the unital associative algebra with generators e, f, k, k^{-1} and relations

$$kek^{-1} = qe, \quad kfk^{-1} = q^{-1}f, \quad (1)$$

$$ef + fe = \frac{k - k^{-1}}{q - q^{-1}}, \quad kk^{-1} = k^{-1}k = 1. \quad (2)$$

The \mathbb{Z}_2 grading $d(e) = d(f) = 1$, $d(k) = 0$ is compatible with the relations, hence has a unique extension to a grading of $\mathcal{U}_q(\text{osp}(1|2))$. We write $\mathcal{U}_q(\text{osp}(1|2)) = \mathcal{U}_q(\text{osp}(1|2))_0 \oplus \mathcal{U}_q(\text{osp}(1|2))_1$ and refer to elements in the first (resp. second) summand as bosons (resp. fermions). As usual $\mathcal{U}_q(\text{osp}(1|2))_0$ is a subalgebra of $\mathcal{U}_q(\text{osp}(1|2))$.

The algebra $\mathcal{U}_q(\text{osp}(1|2))$ has a number of obvious auto-morphisms. For instance the scale change $(e, f, k, k^{-1}) \rightarrow (ae, a^{-1}f, k, k^{-1})$ (where a is a nonzero complex number), the signed permutation $(e, f, k, k^{-1}) \rightarrow (-f, e, k^{-1}, k)$ and combinations thereof induce auto-morphisms.

Of central interest for us is S , the Scasimir operator. It is the boson defined by

$$S = q^{1/2}k - q^{-1/2}k^{-1} - \eta fe \quad (3)$$

where $\eta = (q^{1/2} + q^{-1/2})(q - q^{-1})$.

One can check directly the remarkable fact that

Proposition 1 (Leśniewski [10]) *The operator S anti-commutes with fermions and commutes with bosons. Its square S^2 is nothing but the standard Casimir operator C up to a constant.*

Explicitly,

$$C = qk^2 + q^{-1}k^{-2} + (q - q^{-1})^2(qk + q^{-1}k^{-1})fe - (q - q^{-1})^2(q + 2 + q^{-1})f^2e^2 \quad (4)$$

is related to S by

$$S^2 + 2 = C. \quad (5)$$

The existence of S is not a byproduct of the quantum deformation. Up to an overall factor, S has a well defined classical limit when $q \rightarrow 1$. Set $k = q^h$ and get

$$S_{class} = h - 2fe + \frac{1}{2} = ef - fe + \frac{1}{2}. \quad (6)$$

A normalized version of this operator has first been introduced in [11] where it was used as grading operator on representations. This operator has also been defined in [12], where it is proved that it generates a minimal primitive ideal without Lie algebra analogue since it is not generated by its intersection with the centre. The operator S is also found as part of the image of the quadratic Casimir operator of $\mathcal{U}_q(sl(2))$ by the application defined in [16].

4 Commutations

Recall that q' is the opposite of q . Let m be a positive integer. We define $\varepsilon(m)$ to be 1 if $m = 0, 1 \pmod{4}$ and -1 otherwise.

In the sequel, we shall need explicit (anti)commutations relations between powers of e and f . A recursion argument shows that

$$(q - q^{-1})(f^m e + (-1)^{m-1} e f^m) = f^{m-1} \left(\frac{q'^{-m} - 1}{q'^{-1} - 1} k - \frac{q'^m - 1}{q' - 1} k^{-1} \right). \quad (7)$$

The corresponding equation when the roles of e and f are interchanged can be obtained without computations using the automorphism $(e, f, k, k^{-1}) \rightarrow (-f, e, k^{-1}, k)$.

We adapt a trick used by Kerler [4] for $\mathcal{U}_q(sl(2))$ to get an identity relating the Scasimir operator and k to powers of e and f . The formula

$$\prod_{n=0}^{m-1} (S - q^n q^{1/2} k + q'^{-n} q^{-1/2} k^{-1}) = \varepsilon(m) (-\eta)^m f^m e^m \quad (8)$$

is proved by recursion starting from (3). To go from m to $m + 1$ one multiplies both sides by f on the left, by e on the right and by $-\eta \varepsilon(m) \varepsilon(m + 1)$. The right hand side is what is expected. On the left hand side, f goes through the factors, multiplying k by $-q'$, k^{-1} by $-q'^{-1}$ and S by -1 . Finally it reaches e when (3) is used to eliminate $-\eta f e$. The signs disappear because $(-1)^m \varepsilon(m) \varepsilon(m + 1) = 1$.

5 Foundations

In the sequel, we shall use repeatedly the following

Proposition 2 *The family $\{f^a e^b k^c; a, b \in \mathbb{N}, c \in \mathbb{Z}\}$ is a linear basis of $\mathcal{U}_q(\mathfrak{osp}(1|2))$.*

Applying the (anti-)commutation relations, any element of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ can be written as a linear combination of elements in $\{f^a e^b k^c; a, b \in \mathbb{N}, c \in \mathbb{Z}\}$. That this decomposition is unique is the content of a (super, quantum) analogue of the Poincaré–Birkoff–Witt theorem. To prove it in this special case, one builds $\mathcal{U}_q(\mathfrak{osp}(1|2))$ by two successive Ore extensions of the algebra A_0 of Laurent polynomials in k . Formula (7) plays a crucial role. We do not give the details because they follow closely the proof given in [17], Chap VI, for $\mathcal{U}_q(\mathfrak{sl}(2))$. Chapter I of [17] contains a good introduction to Ore extensions.

We will now prove

Proposition 3 *If q is not a root of unity, the centre of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ is $\mathbb{C}[C]$ (where $C = S^2 + 2$ is the Casimir operator).*

Proof: Consider the commutative subalgebra \mathcal{A} of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ with generators S, k, k^{-1} . First, \mathcal{A} is isomorphic to $\mathbb{C}[k, k^{-1}, S]$. This amounts to say that there is no polynomial relation between k and S . This is because nothing could compensate the monomial of highest degree in S (say m), the only one which produces a term $f^m e^m$ when written in the Poincaré–Birkoff–Witt basis.

Applying formula (8) with $m = \min(a, b)$ to the monomial $f^a e^b k^c$ shows that $\mathcal{U}_q(\mathfrak{osp}(1|2))$ is a free \mathcal{A} -module with basis $1, f, e, f^2, e^2, \dots$. In this basis, it is easy to look for central elements. Until the next section, we assume that q is not a root of unity. The adjoint action of k is diagonal, with eigenvalue $1, q^{-1}, q, q^{-2}, q^2, \dots$ for $1, f, e, f^2, e^2, \dots$. Hence the commutant of k in $\mathcal{U}_q(\mathfrak{osp}(1|2))$ is \mathcal{A} . Now we have to look for the commutant of f and e in \mathcal{A} . For $P(k, S) \in \mathcal{A}$, it is easily shown that $P(k, S)e = eP(qk, -S)$. The commutation condition $P(kq, -S) = P(k, S)$ implies that P has to be an even function of S independent of k . To see this, expand $P(k, S)$ in monomials and identify term by term : a monomial $k^i S^j$ can appear with non zero coefficient only if $q^i (-1)^j = 1$. As q is not a root of unity, i has to vanish and j has to be even. The commutation with f gives the same condition. ■

In the next section, we shall address the question of the structure of the centre when q is a root of unity.

6 Restrictions

For the rest of the paper, l is an integer larger than 2 and q is a primitive l^{th} root of unity. The integer L is the smallest even multiple of l (that is, l if l is even and $2l$ if l is odd). The integer l' is $L/2$ if l is twice an odd integer and L otherwise. Then $q' = -q$ is a primitive l'^{th} root of unity, and $q^L = 1$. We shall now give a complete description of the centre in this case.

Evaluation of (7) for $m = L$ gives

$$f^L e - e f^L = 0. \quad (9)$$

Due to auto-morphisms, the corresponding relation with e and f interchanged is true as well. But $F \equiv f^L$ and $E \equiv e^L$ commute with k . So they are central. We observe again that we do not introduce divided powers, so F and E can be non-zero in representations.

We can now compute the structure of the centre.

Theorem 1 *The centre of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ contains $\mathbb{C}[k^l, k^{-l}, C, F] + \mathbb{C}[k^l, k^{-l}, C, E]$ (this sum is not direct : $\mathbb{C}[k^l, k^{-l}, C, F] \cap \mathbb{C}[k^l, k^{-l}, C, E] = \mathbb{C}[k^l, k^{-l}, C]$).*

- *If l is odd, this is the entire centre.*
- *If l is even, the centre is a free module over $\mathbb{C}[k^l, k^{-l}, C, F] + \mathbb{C}[k^l, k^{-l}, C, E]$ with basis $1, k^{l/2}S$.*

Proof: The commutative (sub-)algebras (of $\mathcal{U}_q(\mathfrak{osp}(1|2))$) $\mathcal{A}[F]$ and $\mathcal{A}[E]$ are free polynomial algebras (the argument given for \mathcal{A} in the previous section still applies). We can use F and E to refine the previously obtained decomposition of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ as an \mathcal{A} -module : $\mathcal{U}_q(\mathfrak{osp}(1|2)) = \mathcal{U}_+ + \mathcal{U}_-$ where \mathcal{U}_+ is the free $\mathcal{A}[F]$ -module with basis $1, f, \dots, f^{L-1}$ and \mathcal{U}_- the free $\mathcal{A}[E]$ -module with basis $1, e, \dots, e^{L-1}$. The sum $\mathcal{U}_+ + \mathcal{U}_-$ is not direct, but $\mathcal{U}_+ \cap \mathcal{U}_- = \mathcal{A}$.

The monomial e^m commutes with k only if m is a multiple of l and with S only if m is even. The same is true for powers of f . Hence the commutant of k and S in $\mathcal{U}_q(\mathfrak{osp}(1|2)) = \mathcal{U}_+ + \mathcal{U}_-$ is $\mathcal{A}[F] + \mathcal{A}[E]$. Again, this sum is not direct and $\mathcal{A}[F] \cap \mathcal{A}[E] = \mathcal{A}$. We adapt the argument of the previous section. Any element of $\mathcal{A}[F] + \mathcal{A}[E]$ can be expanded in powers of k and S . In the expansion of a central element, a monomial $k^i S^j$ can appear with non zero coefficient only if $q^i (-1)^j = 1$. This happens for instance if i is a multiple of l and j is even. If l is odd, this is the only possibility. If l is even, there is another solution, namely, i an odd multiple of $l/2$ and j odd. ■

This gives a simple explicit description of the centre, but there are some drawbacks related to the multiplicative structure. For instance, although E and F are described as such, we still need an expression for FE . This is the purpose of section 9.

Moreover, evaluation of (7) for $m = l'$ gives

$$f^{l'} e + (-1)^{l'-1} e f^{l'} = 0. \quad (10)$$

Due to auto-morphisms, the corresponding relation with e and f interchanged is true as well. Assume l is twice an odd integer, so that $l = L = 2l'$ and l' is odd. Then $e^{l'}$ anti-commutes with k, k^{-1} and S . The above equation shows that it also anti-commutes with f . Analogously, $f^{l'}$ anti-commutes with k, k^{-1}, S and e . So there are unexpected central elements, namely $e^{L/2} f^{L/2} k^{\pm L/2}$. They differ by a factor k^l so only one of those needs to be considered. It will be expressed in the standard description in section 9 as well.

7 Observations

Let us recall some standard facts. The quantum deformation A_q of a semi simple (Lie or super) algebra depends on a complex parameter q . It is generated by raising operators, lowering operators, and Cartan generators. By a choice of Poincaré–Birkoff–Witt basis (a standard choice is to put lowering operators on the left, Cartan generators in the middle and raising operators on the right), we can identify the vector spaces underlying the algebras A_q for different values of q , and it makes sense to talk about a family of elements in A_q depending smoothly on q . A Casimir operator is such a family of central elements. When q is not a root of unity, the Casimirs span the centre of A_q . When q is a root of unity, there is an integer L , simply related to the order of q , such that the L^{th} power of any generator is central. Those new central elements do not satisfy any relation among themselves because such relations would imply relations in the Poincaré–Birkoff–Witt basis (this is automatic for the Lie case, for the super case, one has to be more cautious). At the same time, the Casimirs are not independent from those new generators : there are relations in the centre.

In cases treated up to now (i.e. $\mathcal{U}_q(sl(2))$ in [4], $\mathcal{U}_q(sl(N))$ in [5, 6], $\mathcal{U}_q(sl(2|1))$ in [8]), the following general features have been observed. After a choice of Poincaré–Birkoff–Witt basis, elements of A_q are identified with (non commuting) polynomials, and the notion of substitution is well defined. For instance, if C is a Casimir element, one can replace all the coefficients and generators by their L^{th} power and get a new (central because expressed in terms of central powers of the generators) element $C^{(L)}$. It turns out that $C^{(L)}$ is a polynomial in the Casimirs. This is already remarkable. Moreover, this polynomial has a simple description. It is a generalised Chebychev polynomial : substitution of 0 for the raising and lowering operators gives a restriction involving only Cartan generators, which is enough to compute the desired relation.

This general setting is a little bit abstract. Our point is that it does not work as it stands for $\mathcal{U}_q(osp(1|2))$. However only a minor modification is needed. It is the failure of the general philosophy that motivated us to look for the Scasimir. Once we use the Scasimir instead of the Casimir, the above construction works. So a glance at the section 9 will provide a concrete example of the general facts that we just outlined.

8 Functions

Our subsequent study makes essential use of families of polynomials which we describe now.

If u is an indeterminate and m a positive integer, we claim that $u^m + (-u^{-1})^m$ is a monic polynomial of order n in $S = u - u^{-1}$ of parity $(-1)^m$, in fact a Chebychev polynomial. This comes from trigonometric identities, but we prefer to consider the generating function

$$\sum_{m \geq 0} t^m (u^m + (-u^{-1})^m) = \frac{1}{1 - tu} + \frac{1}{1 + tu^{-1}} = \frac{2 - tS}{1 - tS - t^2} = \sum_{m \geq 0} t^m P_m(S). \quad (11)$$

The polynomial $P_m(S)$ has the expected properties. One checks that $P_2(S) = S^2 + 2$. Then $P_{2m}(S)$ is a polynomial in $C = S^2 + 2$. In fact

$$\sum_{m \geq 0} t^m P_{2m}(S) = \frac{2 - tC}{1 - tC + t^2} = \sum_{m \geq 0} t^m Q_m(C). \quad (12)$$

Comparison of the generating functions shows that

$$P_m(iS) = i^m Q_m(S) \quad Q_m(iS) = i^m P_m(S). \quad (13)$$

Moreover the definition of the polynomials P_m and Q_m in terms of u shows that

$$Q_m(C) = P_m(S)^2 + 2(-1)^{m+1}. \quad (14)$$

For analogous reasons, $P_{2m+1}(S)/S$ is also a polynomial in C . In fact

$$\sum_{m \geq 0} t^m P_{2m+1}(S)/S = \frac{1 + t}{1 - tC + t^2} = \sum_{m \geq 0} t^m R_m(C). \quad (15)$$

9 Relations

The clue is to consider equation (8) for $m = l'$.

$$\prod_{n=0}^{l'-1} (S - q'^n q^{1/2} k + q'^{-n} q^{-1/2} k^{-1}) = \varepsilon(l') (-\eta)^{l'} f^{l'} e^{l'}. \quad (16)$$

The nice feature is that inside the product q'^n runs over all l' th roots of unity. The left hand side involves only S , k and k^{-1} , which commute among themselves. There is a standard way to simplify the product. We introduce commuting variables u and v and compute

$$\begin{aligned} \prod_{n=0}^{l'-1} (u - u^{-1} - q'^n v + q'^{-n} v^{-1}) &= \prod_{n=0}^{l'-1} u^{-1} (u - q'^n v) (u + q'^{-n} v^{-1}) \\ &= u^{-l'} \prod_{n=0}^{l'-1} (u - q'^n v) \prod_{n=0}^{l'-1} (u + q'^{-n} v^{-1}) \\ &= u^{-l'} (u^{l'} - v^{l'}) (u^{l'} - (-v^{-1})^{l'}) \\ &= u^{l'} + (-u^{-1})^{l'} - v^{l'} - (-v^{-1})^{l'}. \end{aligned}$$

In this identity we set $S = u - u^{-1}$ and $v = q^{1/2} k$. On the right hand side we recognise the Chebychev polynomials from section 8. Hence (16) becomes

$$P_{l'}(S) = q'^{l'/2} k^{l'} + (-1)^{l'} q^{-l'/2} k^{-l'} + \varepsilon(l') (-\eta)^{l'} f^{l'} e^{l'}. \quad (17)$$

This gives a relation between S and the l' th powers of the other generators. We can now check the properties announced in the previous section. The right hand side is obtained essentially by raising to the l' th power the terms in the equation defining S . The polynomial in S on the left hand side is fixed by its value in the quotient where $e = f = 0$ so that only Cartan generators survive.

From this it is clear how to get relations in the centre. We have to distinguish several cases.

Proposition 4 *If l is not twice an odd integer, $l' = L$ is even, and (17) is a relation in the centre :*

$$(-1)^{L/2} Q_{L/2}(C) = -k^L - k^{-L} + \eta^L f^L e^L. \quad (18)$$

As observed in section 6, if l is twice an odd integer, then $l = L = 2l'$ and l' is odd. Then $e^{l'}$ and $f^{l'}$ anti-commute among themselves and with k , k^{-1} and S . Hence (17) is a relation in the centre. Multiplying by $k^{l'}$ on both sides gives the relation in the centre written below (19). Another relation in the centre (20) expresses $Q_{L/2}(C) = P_{l'}(S)^2 + 2$ in terms of central elements. Using (19) this gives a formula for $f^L e^L$ in the standard basis of the centre (21). To summarise

Proposition 5 *If l is twice an odd integer, the following relations in the centre hold:*

$$S k^{L/2} R_{\frac{L-2}{4}}(C) = q^{L/4} k^L - q^{-L/4} + (-1)^{\frac{L+2}{4}} \eta^{L/2} f^{L/2} e^{L/2} k^{L/2}, \quad (19)$$

$$Q_{L/2}(C) = -k^L - k^{-L} + 2(-1)^{\frac{L+2}{4}} q^{L/4} \eta^{L/2} f^{L/2} e^{L/2} (k^{L/2} + k^{-L/2}) - \eta^L f^L e^L, \quad (20)$$

$$Q_{L/2}(C) = k^L + k^{-L} + 4 + 2q^{L/4} S (k^{L/2} + k^{-L/2}) R_{\frac{L-2}{4}}(C) - \eta^L f^L e^L. \quad (21)$$

10 Representations

The representation theory of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ has already been studied by several authors [18, 19, 20]. It seems to us, however, that a complete classification of them, including both periodic and nilpotent ones did not exist in the case q a root of unity. Our point is also to illustrate the use of the Scasimir in such a classification, as in [12] for the classical case.

First we list some families of representations of $\mathcal{U}_q(\mathfrak{osp}(1|2))$, show that they are irreducible and give the possible isomorphisms among them. After that we show that any irreducible representation appears in our list.

Let V be a vector space with basis $|0\rangle, \dots, |L-1\rangle$, with the convention that $|L\rangle = |0\rangle$. We define operators Q, U, P acting on V by

$$Q|m\rangle = q^{-m}|m\rangle \quad U|m\rangle = (-1)^m|m\rangle \quad P|m\rangle = |m+1\rangle. \quad (22)$$

We endow V with several structures of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ -modules.

- The module $M_+(\lambda, \phi, \sigma)$ depends on three complex parameters, the first two are nonzero. As a vector space $M_+(\lambda, \phi, \sigma)$ is V . We set

$$k = \lambda Q \quad f = \phi P \quad e = \frac{1}{\eta\phi} P^{-1} (q^{1/2} \lambda Q - q^{-1/2} \lambda^{-1} Q^{-1} - \sigma U). \quad (23)$$

By definition, σU anti-commutes with f and e , but commutes with k , and the definition of e gives it the status of the Scasimir operator, so the structure equations of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ are trivially satisfied.

Conjugation by Q , P and U shows that

$$M_+(\lambda, \phi, \sigma) \cong M_+(\lambda, q^{-1}\phi, \sigma) \cong M_+(q\lambda, \phi, -\sigma) \cong M_+(\lambda, -\phi, \sigma). \quad (24)$$

The value of the central element e^L can be computed in terms of the parameters of $M_+(\lambda, \phi, \sigma)$. We write this value as $\bar{\epsilon}^L$ for a certain $\bar{\epsilon}$.

- The module $M_-(\lambda, \epsilon, \sigma)$ depends on three complex parameters, the first two are non-zero. As a vector space $M_-(\lambda, \epsilon, \sigma)$ is V . We set

$$k = \lambda Q \quad e = \epsilon P^{-1} \quad f = \frac{1}{\eta \epsilon} \left(q^{1/2} \lambda Q - q^{-1/2} \lambda^{-1} Q^{-1} - \sigma U \right) P. \quad (25)$$

Again, for analogous reasons, the structure equations of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ are trivially satisfied, and there are equivalences.

If we choose the parameters of $M_+(\lambda, \phi, \sigma)$ and $M_-(\lambda, \epsilon, \sigma)$ such that $\bar{\epsilon} = \epsilon$, then $M_+(\lambda, \phi, \sigma) \cong M_-(\lambda, \epsilon, \sigma)$. This is because we can change basis in $M_+(\lambda, \phi, \sigma)$ by setting $|\overline{m}\rangle = \epsilon^m e^{-m} |m\rangle$. In this new basis, e, k, S act like on $M_-(\lambda, \epsilon, \sigma)$, and there is no freedom to define f .

We claim that unless l is odd and $\sigma = 0$ the above representations are irreducible. This is because any submodule would contain a common eigenvector of k and S . Assume we are dealing with a module of type M_+ for example. The action of the invertible operator f will create L non-zero vectors which are distinguished by the eigenvalues of k and S , hence are linearly independent.

In case l is odd and $\sigma = 0$, the module is irreducible if considered as graded-module, for the same reasons as above (the gradation playing the role played by S when $\sigma \neq 0$).

In contrast with the classical case [12] and the case q is not a root of unity, an ungraded finite dimensional simple module cannot always be endowed with a gradation, as shown by the following. In case l is odd and $\sigma = 0$, the (ungraded) module is not irreducible. Assume again we are dealing with a module of type M_+ . The invertible operator f^l commutes with P and Q (or k), hence also with e , but is not scalar. A non trivial eigenspace of f^l is a subrepresentation. We describe explicitly but without details the corresponding representations. Consider a vector space V' with basis $|0\rangle, \dots, |l-1\rangle$, with the convention that $|l\rangle = |0\rangle$. We define operators Q, P (but not U) acting on V' by formulæ analogous to (22). Then

- The module $M_+(\lambda, \phi)$ depends on two non-zero complex parameters. As a vector space $M_+(\lambda, \phi)$ is V' . The action of k, f and e is defined by formulæ (23) with $\sigma = 0$.
- The module $M_-(\lambda, \epsilon)$ depends on two non-zero complex parameters. As a vector space $M_-(\lambda, \epsilon)$ is V' . The action of k, e and f is defined by formulæ (25) with $\sigma = 0$.

The equivalences between those modules follow from the previous equivalences. Those modules are irreducible.

Now, we describe another type of representations $M_d(\lambda)$ where both e and f are nilpotent. Let V_d be a vector space with basis $|0\rangle, \dots, |d-1\rangle$, with the convention that $|-1\rangle = |d\rangle = 0$. We define operators Q , U and P as in (22) but note that P is not invertible anymore ($P|d-1\rangle = 0$). We also define P' by $P'|m\rangle = |m-1\rangle$. Then we set

$$k = \lambda Q \quad f = P \quad e = \frac{1}{\eta} P' (q^{1/2} \lambda Q - q^{-1/2} \lambda^{-1} Q^{-1} - \sigma U). \quad (26)$$

For the time being, λ and σ are arbitrary complex parameters. This time, one has to check explicitly the structure equations of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ because $f = P$ is not invertible, so P' is not an inverse. The only nontrivial check is that $ef + fe = \frac{k-k^{-1}}{q-q^{-1}}$ at the boundaries $|0\rangle$ and $|d-1\rangle$. One finds

$$q^{1/2} \lambda - q^{-1/2} \lambda^{-1} = \sigma = q^{1/2} q'^{-d} \lambda - q^{-1/2} q'^d \lambda^{-1}. \quad (27)$$

These equations give σ in terms of λ and

$$(q'^d - 1)(\lambda^2 - q'^{d-1}) = 0. \quad (28)$$

Unless $q'^d = 1$ (that is unless d is a multiple of l'), this is a quantisation condition for λ . The presence of $q' = -q$ (instead of q) in (28) explains why the even dimensional modules have no classical limit.

Now we have to check irreducibility. By construction the basis vectors of V_d are eigenvectors of k and S and $|d-1\rangle$ is annihilated by f . From section 5 we know the structure of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ as an \mathcal{A} -module. It implies that acting with powers of e on $|d-1\rangle$ generates a submodule. So the representation can be irreducible only if e^{d-1} does not act as 0. The converse is also true because e is nilpotent on V_d , so has a kernel in any submodule. Hence V_d is irreducible if and only if $e^{d-1}|d-1\rangle \neq 0$, which is equivalent to $f^{d-1}e^{d-1}|d-1\rangle \neq 0$. This last condition is easy to check using (8) for $m = d-1$. A simple computation gives the irreducibility criterion :

$$\prod_{n=1}^{d-1} (q'^n - 1)(q^{1/2-d} \lambda + q^{d-1/2} \lambda^{-1} q'^{-n}) \neq 0. \quad (29)$$

Equivalently, V_d is irreducible if either $d < l'$ or $d = l'$ and $q'\lambda^2$ is not a nontrivial power of q' .

It is now easy to show

Theorem 2 *The finite dimensional irreducible representations of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ are:*

- *The f -periodic modules $M_+(\lambda, \phi, \sigma)$ (23) of dimension L (take instead $M_+(\lambda, \phi)$ of dimension l if l is odd and $\sigma = 0$)*

- The e -periodic modules $M_-(\lambda, \epsilon, \sigma)$ (25) of dimension L (take instead $M_-(\lambda, \epsilon)$ of dimension l if l is odd and $\sigma = 0$)
- The nilpotent modules $M_d(\lambda)$, of dimension $1 \leq d \leq l'$, with the conditions (28) and (29).

Modules of type M_+ and M_- are equivalent if, and only if, they share the action of the central elements.

Proof: What remains to prove is that every irreducible finite dimensional representation of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ appears in the above list

The proof goes as follows. Let V be an irreducible finite dimensional representation space of $\mathcal{U}_q(\mathfrak{osp}(1|2))$. The operators k and S can be simultaneously diagonalised on V . The reason is that they commute, so they have a common eigenvector. As a consequence of the commutations relations of the generators of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ with k and S , monomials in e, f, k, k^{-1} applied to this vector are clearly either 0 or again eigenstates k and S . Those monomials span a subrepresentation which has to be V itself by irreducibility. So we endow V with a basis consisting of eigenvectors of k and S .

Now we distinguish three cases. First, suppose f is invertible on V . Then knowing the action of k , S and f fixes the action of e uniquely by $\eta e = f^{-1}(q^{1/2}k - q^{-1/2}k^{-1} - S)$. As f^L is central, starting from a common eigenvector of k and S and acting with f one builds a subrepresentation of dimension at most L , which has to be V itself. Unless l is twice an odd integer and $S = 0$, V has to be L -dimensional because $1, f, \dots, f^{L-1}$ are distinguished by the eigenvalue of k and S . If l is twice an odd integer and $S = 0$ we can diagonalise $f^{l'}$ and k . So we always end-up with an irreducible representation of type M_+ . The second case is when e is invertible. The same line of arguments leads to an irreducible representation of type M_- . The third case is when neither f nor e is invertible. We start from an eigenvector of k and S annihilated by e and call it the highest weight vector. The structure of $\mathcal{U}_q(\mathfrak{osp}(1|2))$ as an \mathcal{A} -module implies that powers of f acting on the highest weight build a subrepresentation which has to be V itself. The largest non-vanishing power, say $d - 1$, of f acting on the highest weight gives a lowest weight. For the same reasons, powers of e acting on the lowest weight build a subrepresentation which has to be V itself. The formula relating k, f, e to S shows that V has to be equivalent to a representation on V_d listed above. We have already given a criterion for irreducibility for those.

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